THE RADIATION OF AN ISOTHERMAL SPHERE WITH CONSIDERATION OF SCATTERING

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We have solved the problem of the radiation from an isothermal sphere with a spherical scattering indicatrix. We demonstrate that the emissivity of the sphere and of the plane layer, with consideration of scattering, can be approximately presented by a single function of the product resulting from the multiplication of the attenuation factor by the geometric characteristic of the radiating volume.

As is well known, the Schwarzschild-Schuster and Eddington approximations for the problems of radiative heat exchange are not sufficiently accurate. Below we will employ the Bubnov-Galerkin method to examine the problem of radiation from an isothermal sphere with a spherical scattering indicatrix. To refine the solution, we will use the method of iterations. We used this method to calculate the radiative heat exchange in a plane layer in [1]. The equation for radiation transfer in the case under consideration [2] has the form

$$\mu \frac{\partial I}{K\partial r} + \frac{1-\mu^2}{Kr} \frac{\partial I}{\partial \mu} + I = \frac{\lambda}{2} \int_{-1}^{1} I d\mu + 2(1-\lambda) \tag{1}$$

with the boundary condition $I(\tau_0, \mu < 0) = 0$, which indicates that the sphere is bounded by a vacuum or by an absolutely black nonradiating surface. The attenuation factor k and the scattering coefficient at a ratio of the attenuation factor λ will be held to be constant. The dimensionless flow of energy at the boundary of the sphere is expressed in the form

$$Q(\tau_0) = \int_0^1 \mu I(\tau_0, \mu) d\mu, \tag{2}$$

where τ_0 is the product of the attenuation factor by the radius of the sphere. The source function of Eq. (1) is written so that $Q(\tau_0)$ is equal to the emissivity.

Let us integrate Eq. (1) with respect to μ in the limits $(-1, 1)$. This yields

$$\frac{dQ}{d\tau} + \frac{2}{\tau} Q = -(1-\lambda) \int_{-1}^{1} I(\tau, \mu) d\mu + 4(1-\lambda). \tag{3}$$

From the symmetry of the problem, the boundary condition for this equation is

$$Q(0) = 0. \tag{4}$$

Solving Eq. (3) with boundary condition (4) formally, we have

$$Q = \frac{1-\lambda}{\tau^2} \int_0^{\tau} \tau^2 \left[4 - \int_{-1}^{1} I(\tau, \mu) d\mu \right] d\tau. \tag{5}$$

If we introduce the function

$$B(\tau) = \frac{\lambda}{2} \int_{-1}^{1} I(\tau, \mu) d\mu + 2(1-\lambda), \tag{6}$$

as is well known [2], for this function the following integral equation is valid:

$$\tau B(\tau) = \frac{\lambda}{2} \int_0^{\tau_0} \tau' B(\tau') [E_1 |\tau - \tau'| - E_1(\tau + \tau')] d\tau' + 2(1-\lambda)\tau. \tag{7}$$

Solving this equation by the Bubnov-Galerkin method, assuming $B \equiv B_0 = \text{const}$, we have

$$B_0 = \frac{2(1-\lambda)}{1 - \lambda\left\{1 - \frac{1}{\tau_0^2}\left[\tau_0\left(\frac{1}{2} - 2E_3(\tau_0) + E_3(2\tau_0)\right) + \left(\frac{1}{3} - 2E_4(\tau_0) + E_4(2\tau_0)\right)\right]\right\}}, \tag{8}$$

where E_n is an integral-exponential function of n-th order. A rather detailed table of these functions—to the 3rd order—is found in references [3, 7]. The most detailed table for E_1 is found in [4]. Integral-exponential functions of higher orders can be found from the familiar recurrence formula

$$nE_{n+1}(x) = e^{-x} - xE_n(x),$$

and from (1), (2), and (8) it is not difficult to find the emissivity of the isothermal sphere

$$Q_0(\tau_0) = \frac{B_0}{2} \varepsilon_0(\tau_0), \qquad (9)$$

where $\varepsilon_0(\tau_0)$ is the emissivity of the nonscattering sphere of optical thickness τ_0. As is well known [6], for ε_0 the following relationship holds:

$$\varepsilon_0(\tau_0) = 1 - \frac{1 - e^{-2\tau_0}(1 + 2\tau_0)}{2\tau_0^2}. \qquad (10)$$

Formula (9) may be used only for small optical thicknesses $\tau_0 \leqslant 1$. It is the more exact, the smaller τ_0. To refine the solution, let us use the method of iterations. Having substituted B_0 in place of B in the right-hand member of Eq. (7), we find that

$$B(\tau) = 2(1-\lambda) + \frac{\lambda}{2\tau} B_0 \{2\tau - \tau_0 [E_2(\tau_0 - \tau) - E_2(\tau_0 + \tau)] - [E_3(\tau_0 - \tau) - E_3(\tau_0 + \tau)]\}. \qquad (11)$$

It follows from (6) and (11) that

$$\int_{-1}^{1} I d\mu = \frac{B_0}{\tau} \{2\tau - \tau_0 [E_2(\tau_0 - \tau) - E_2(\tau_0 + \tau)] - [E_3(\tau_0 - \tau) - E_3(\tau_0 + \tau)]\}.$$

From (5) we find that

$$Q(\tau_0) = (1-\lambda)\left[\frac{4}{3}\tau_0 - B_0 \psi(\tau_0)\right], \qquad (12)$$

where

$$\psi(\tau_0) = \frac{2}{3}\tau_0 - \frac{1}{2} - E_3(2\tau_0) + \frac{1}{\tau_0^2}\left(\frac{1}{4} - 2\tau_0 E_4(2\tau_0) - E_5(2\tau_0)\right).$$

It is not difficult to prove, using the recurrence formula for E_n, that

$$\psi(\tau_0) = \frac{2}{3}\tau_0 - \frac{1}{2}\varepsilon_0(\tau_0). \qquad (13)$$

If $\tau_0 \gg (1/1-\lambda)$, then $B_0 = 2 - (\lambda/(1-\lambda)\tau_0)$. Consequently, when $\lambda \neq 1$, and $\tau_0 \to \infty$, we have

$$Q(\tau_0) = 1 - \frac{\lambda}{3}, \qquad (14)$$

i.e., for the albedo of a semi-infinite medium we can write the relationship

$$R_\infty \approx \frac{\lambda}{3}. \qquad (15)$$

To check on the accuracy of formula (12), we compare (15) in Table 1 with the results from the calculation of R_∞ carried out by V. V. Sobolev [5], as well as with the data found from the results of reference [8]. To find R_∞ from the data of [8] we used the formula for the radiant flux incident on the wall of a plane isothermal layer bounded by isotropically reflecting surfaces; this formula was derived in reference [9]. Since (12) is the more exact, the smaller τ_0, we can expect from an analysis of Table 1 that the results from the calculation of the emissivity on the basis of formula (12) will be satisfactory when $\lambda \leqslant 0.7$. If $\lambda > 0.7$, the calculation can be satisfactory only if the optical thicknesses are not too great.

It was demonstrated in [6] by A. S. Nevskii that if we introduce the parameter

$$\delta = \frac{4V}{F}, \qquad (16)$$

referred to as the geometric characteristic of the radiating volume, for a nonscattering infinite cylinder, sphere, or plane layer the emissivity is expressed, in approximate terms, by a universal function of the product $K\delta$. Table 2 shows the emissivity of a plane layer and of a sphere with a spherical scattering indicatrix as functions of the parameter $K\delta$ for λ. We used

Table 1

The Albedo of a Semi-infinite Medium

λ	$\lambda/3$	R_∞ according to the data [5]	R_∞ from [8]
0.1	0.033	—	0.024
0.2	0.067	—	0.047
0.3	0.100	—	0.076
0.4	0.133	0.11	0.114
0.5	0.167	0.15	0.149
0.6	0.200	0.19	0.195
0.7	0.233	0.26	—
0.8	0.267	0.34	—
0.9	0.300	0.48	—

Table 2

The Emissivity of an Isothermal Layer and of an Isothermal Sphere as a Function of the Parameter $K\delta$ (the scattering indicatrix is spherical)

$K\delta$	$\lambda = 0.2$		$\lambda = 0.5$		$\lambda = 0.8$	
	plane layer	sphere	plane layer	sphere	plane layer	sphere
0.2	143	155	093	094	039	040
0.4	257	270	173	177	075	076
0.6	350	380	243	253	110	112
0.8	427	455	305	316	142	145
1.0	492	524	359	372	173	177
1.2	548	580	408	425	202	209
1.4	596	630	451	468	229	235
1.6	637	670	490	504	255	260
1.8	673	704	525	537	279	284
2.0	706	734	556	572	303	305
2.2	734	760	584	599	324	330
2.4	759	790	610	625	345	351

the Ivon [1] method to calculate the emissivity for the plane layer, and we calculated the emissivity of the sphere from formula (12). We can conclude from Table 2 that the emissivities of the sphere and of the plane layer, with consideration of scattering, can be presented, in approximate terms, by a single function of the product resulting from the multiplication of the attenuation factor by the geometric characteristic of the radiating volume. Since the shape of the infinite cylinder occupies an intermediate position between the shape of the plane layer and that of the sphere, it is to be expected that the emissivity of the cylinder as a function of $K\delta$ will occupy an intermediate position between the emissivity of the sphere and that of the plane layer.

NOTATION

I is the intensity of radiation; μ is the cosine angle between the sphere radius and the radiation direction; λ is the scattering-to-attenuation ratio; K is the attenuation factor; r is the modulus of the radius vector from the center of the sphere; τ_0 is the optical thickness of the sphere; Q is the dimensionless energy flux; ε_0 is the emissivity of the nonscattering sphere; E_r is the integral power function of n-th order; R_∞ is the albedo of semi-infinite medium; V is the volume of the medium; δ is the geometric parameter of the radiating volume; F is the surface area of the radiating volume.

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